Decomposing a Multiobjective Optimization Problem into a Number of Reduced-Dimension Multiobjective Subproblems Using Tomographic Scanning

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***Abstract*—In this paper, we design a novel method to handle multi- and many-objective optimization problem. The proposed method adopts the idea of tomographic scanning in medical imaging to decompose the objective space into a combination of many tomographic maps to reduce the dimension of objectives incrementally. Moreover, subpopulations belonging to different tomographic maps can help each other in evolving the optimal results. We compared the performance of the proposed algorithm with some classical algorithms such as NSGA-II and MOEA/D-TCH and their state-of-the-art variants including MOEA/D-DE, NSGA-III and MOEA/D-PBI. The experimental results demonstrate that the proposed method significantly outperforms MOEA/D-TCH, MOEA/D-DE and NSGA-II, and is very competitive with MOEA/D-PBI and NSGA-III in terms of convergence speed.**

***Keywords*-Many-Objective Optimization, Reduce dimensionality, Decomposition**

I. Introduction

Many of real-world problem that have the number of objectives or criterions more than one, which stated as multi-objective optimization problem(MOP). A MOP can be defined as follow:

$$\begin{aligned} &minimize \qquad F(X) = (f_1(X), ..., f_m(X))^T \\ &subject\ to \qquad X \in \Omega \end{aligned} \tag{1}$$

Where Ω is decision space, $F(X)$ is a vector of objective and m is the number of objective with the problem. Meanwhile, these objectives usually conflict with each other. In general, hence, there have no point in space Ω which can make each objective to reach the optimal value simultaneously. Unlike single objective optimization problems, which generally just have single optimal solution, in MOP, there have a set of optimal solutions instead of only solution. Such a set called *Pareto Set*(PS), mapping the *Pareto Set*(PS)to the objective space obtained *Pareto Front*(PF).

In the past decade, multi-objective evolutionary algorithms(MOEAs) have been widely used for solving MOPs with two- or three-objective, which concerned with performances of convergence and diversity for algorithm. Some state-of-the-art MOEAs have been proposed, such as dominance based NSGA-II[1] and decomposition-based method MOEA/D[2]. However, with the increase of practical demands, MOPs involve four or more objectives. Many of difficulties appear for current MOEAs' designer. The primary difficulty is that pressure of selection decreased with dimensionality increasing. The more details present in Section II.

During this period of time, many researchers also try to solve a more higher dimensionality MOP using current evolutionary multi-objective optimization(EMO). In this paper, we review some of previous efforts to deal with many-objective optimization problem(MaOP) that have four or more objectives, which include visualization[3], reduced-dimensionality using principal component analysis[4],[5], preference-based using reference-points[6]. However, there still have certain performances been worth to develop.

In this paper, we compared our proposed algorithm NSGA/TD with some state-of-the-art MOEAs which are MOEA/D-TCH[2], MOEA/D-PBI[2], MOEA/D-DE[7]. The experimental result have shown at latter section.

The remainder of this paper organized as follows. In section II, we discuss difficulties in MaOP and some of methods presented for solving MaOP in sectionIV. Therefore, we outline our proposed method in detail. Result of the DTLZ test problem[8] are shown in section IV. Finally, the conclusion of this paper are drawn in section V.

II. Difficulties with Many-Objective Optimization Problems and current EMO for Many-Objective Optimization Problems

In general, many-objective optimization problems have four or more objectives to optimize simultaneously. With increasing dimensionality of objectives, traditional MOEAs are becoming inefficient when solving many-objective optimization problem (MaOP). A series of difficulties in many-objective optimization problems have been stated in[6],[9], two primary difficulties are presented as the following:

1) Difficulty in convergence, that is, while the dimensionality of objectives increased, almost of all individuals do not dominate each other and difficult to generate new non-dominated solution efficiently. Thereby, pressure of evolutionary selection will decrease significantly. Above can slow down efficient of search process or search process will be stagnant from perspective of convergence.
2) With increasing objective, leading to aggravation of conflict between convergence and diversity [10].

From above, the most primary difficulty in high-dimensionality objective is decreasing of selection pressure. There have proposed a lot of method for enhancing the selection pressure, balancing convergence and diversity. These have four types of method for dealing with MOEAs:

1) Modified traditional definition of Pareto dominance and proposed new principle of dominance for strengthening selection pressure, such as epsilon dominance (ε-*dominance*), L-Optimality[11], Fuzzy-domination[12] and level sorting based on preference[13], etc.
2) To combine classical principle of both Pareto-dominated and convergence-based. This method adopts dominated principle to sort firstly, then selects solutions by convergence principle, such as GrEA[14].
3) To design innovation mechanism that based on metrics performance. There has three algorithm, IBEA[15], SMS-EMOA[16], HypE[17]. IBEA adopt pre-defined optimization problem to measure distribution for each solution and that SMS-EMOA and HypeE select solution by value of Hypervolume.
4) To decompose high-dimension objective space for reduced the dimensionality and that searching based on referent-point or pre-define direction, classical methods include MOEA/D[2], MOEA/D-M2M[18], NSGA-III [6]and principal component analysis(PCA-NSGA-II[4],L-PCA[5]), etc.

III. Main Idea and Algorithm

The proposed idea concerned providing a decomposition method for solving MaOP, the main different is gradually decomposed an MaOP to many two- or three-objectives MOP.

As mentioned before, many-objective optimization problems have M objectives that are often four or more and the dimensionality of PF which is an m-dimension hypersurface(if the hyper-surface was linear, then called hyperplane). Loosely speaking, there have two type of problems in many-objective optimization problem, Hard-MaOP and Soft-MaOP, by relationship between dimensionality of objectives and PF.

- **Hard-MaOP**: If $m = M$, the dimensionality of problem is equal to dimensionality of PF. In this situation, conflict exists between objectives which have no redundancy or relative. In the case of the DTLZ2(three objectives) [8], the PF shown at Figure 1.
- **Soft-MaOP**: If $m < M$, the dimensionality of PF is lower than dimensionality of problem. There have some redundancy between objectives. In the case of the DTLZ5(three objectives)[8], the PF shows as Figure 1.

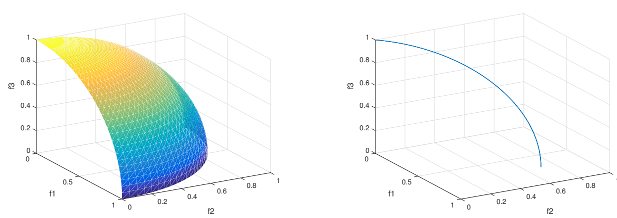

Fig. 1. Pareto Front of MOP.(Left: DTLZ2, Right: DTLZ5)

As analyzed in PCA-NSGA-II[4], when redundancy existed within MOP, we can obtain some relativity between objectives from the analyzed result, then we can reduce the dimensionality of objectives. In the case of DTLZ5, f_1 is linear relative with f_2, so the number of objectives can reduce from three to two(f_3,f_2 or f_3,f_1) and pressure of selection will increase.

To be inspired by PCA-NSGA-II, we proposed a method tomographical decomposition(TD) to deal MaOP. In similar, if we can decompose a Hard-MaOP to many Soft-MaOP, the problem can be get more redundancy and the pressure of selection will increase with redundancy increased. For Firstly, starting in three-objective optimization problem, for establishing the faultage to use a hyper-plane which crosses axe of f_3 and intersect a curve C_i, which curve is a part of PF. If we used enough hyper-plane, the whole PF will be present by the set of curves C_i. Such each hyper-plane is a subproblem, each curve is PF of the subproblem. And each subproblem has been related by a population to search optimal solutions using traditional MOEAs.

Then, there has given faultage definition and subproblem decomposition as the following:

$$\begin{aligned} f_2 &= k f_1 \\ k &\in \mathbb{R} \end{aligned} \tag{2}$$

where k is a factor that decided the position of faultage. We suggested individuals that have same behaviors in same faultage, which show at decision variables with some particular mode or function, such as certain decision variable is decided individual at which faultage. In the case of a series of test problem DTLZ, we can obtain certain relationship between space of objective and decision for appointed value of k. For example, in DTLZ1, for satisfying condition within Eq.(2), we can obtain relationship $x_2 = \frac{1}{k+1}$ through solving Eq.(2), and

Problem Name	Relevant Variable and Relationship
DTLZ1	$x_2 = \frac{1}{k+1}$
DTLZ2	$x_{M-1} = \arctan k \times \frac{\pi}{2}$
DTLZ3	$x_{M-1} = \arctan k \times \frac{\pi}{2}$
DTLZ4	$x_{M-1} = \sqrt[100]{\arctan k \times \frac{\pi}{2}}$

TABLE I
RELATIONSHIP BEWTEEN FAULTAGE AND INDIVIDUAL VARIABLE

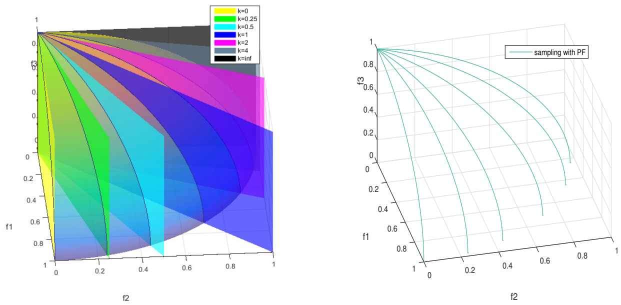

Fig. 2. PF of DTLZ2 and DTLZ5 test problem, (Left: DTLZ2, Right: DTLZ5)

we can know that other decision variables are irrelevant with position of faultage. In similar to DTLZ2-DTLZ4, we can also get certain relationship within decision variables. Hence, factor k can determine the position of faultage and individuals.In our work, we have the relationship in DTLZ1-DTLZ4 shown at Table I. By this way, we can obtain key variables to decide which faultage individual generated.

IV. EXPERIMENTAL DESIGN AND EXPERIMENTAL RESULT

A. Experimental Setting

In order to evaluate the performance of proposed method in Section III, we combined the method with NSGA-II and then studied the experimental results on DTLZ1-DTLZ4[8]. Thirty independent runs with the four algorithms are conducted. The experimental parameter set as follows.

- The mutation probability $P_m = 1/n$ (n is the number of decision variables). For the DE operator, we set $CR = 1.0$ and $F = 0.5$ as recommended in[19].
- Each population size: $N = 30$. Population number: $Popn = 5$.
- Number of runs and stopping condition: Each algorithm runs 30 times independently on each test problems. The algorithm stops until $300,000$ function evaluations.

B. Performance Metric

In our work, performance of many-objective evolutionary algorithm is evaluated in two aspects convergence and distribution. Convergence describes the closenfess of the obtained solutions to the true Pareto front. Distribution depicts diversity of obtained solutions in objective space. Two metrics were chosen - inverted generation distance (*IGD*)[20] and hypervolume(*HV*)[21]. Detailed definitions are given as follows.

Inverted Generational Distance (*IGD*):

Let R^* is the true Pareto front set, P is a set of obtained solutions achieved by evolutionary multi-objective algorithm. *IGD* metric denotes the Euclidean distance between R^* and P. It is defined as follows:

$$
\begin{aligned}
IGD(R^*, P) &= \frac{\sum_{y^* \in R^*} d(y^*, P)}{|R^*|} \\
d(y^*, P) &= min \left\{ \sqrt{\sum_{i=1}^{m} (y_i^* - y_i)^2} \right\}
\end{aligned}
\tag{3}
$$

Where M is the number of objectives, $|R^*|$ denotes the size of set R^*, $d(y^*, P)$ denotes the minimum Euclidean distance between y^* and P. *IGD* metric can present the convergence and diversity simultaneously. The smaller *IGD* metric means the better performance.

Hypervolume (*HV*):

HV simultaneously considers the distribution of the obtained Pareto front P and its vicinity to the true Pareto front. *HV* is defined as the volume enclosed by P and the reference vector $r = (r_1, r_2, ..., r_m)$. *HV* can be defined as:

$$HV(R) = \bigcup_{i \in R} voli \tag{4}$$

Here, $vol(i)$ denotes the volume enclosed by solution $i \in P$ and the reference vector r. The maximum value of each objective in the true Pareto front set gives the value of each dimension of the reference point r, and thus constructs the reference point.

C. Experimental Result

In order to demonstrate the effectiveness of the proposed decomposition method, we first compared NSGA/TD with NSGA-II. The final populations with the best IGD and HV metric in 30 independent runs for DTLZ1-DTLZ4 compared with NSGA-II are shown in Figure 3. As shown result, NSGA/TD is better than NSGA-II for DTLZ1-DTLZ4 in metrics HV and IGD. The variation of IGD metric value with NSGA-II and NSGA/TD are shown in Figure 4. The figure shown the convergence speed of NSGA/TD is significantly better than NSGA-II. Furthermore, the result that compared with current state of the art MOEAs which include MOEA/D-PBI, MOEA/D-DE, MOEA/D-TCH and NSGA-III are shown at Table VI. As shown result, NSGA/TD is better than other four algorithms for DTLZ1 and better than MOEA/D-TCH and MOEA/D-DE for DTLZ2-DTLZ4 in IGD metric. The experimental results shown NSGA/TD significantly outperformed NSGA-II in DTLZ1-DTLZ4 test problem and quite comparable with NSGA-III, MOEA/D-PBI, MOEA/D-TCH and MOEA/D-DE.

V. CONCLUSION

In this paper, we have proposed tomographic decomposition and algorithm NSGA/TD. To design experiment to demonstrate the efficiency of tomographic decomposition to handle 3-objective MOP and also compared with some state-of-the-art algorithms. The result shown method of tomographic decomposition is effective for accumulating speed of convergence

TABLE II
BEST, MEDIAN, AND WORST IGD METRIC VALUES OBTAINED FOR NSGA/TD AND NSGA-II ON 3-OBJECTIVE DTLZ1, DTLZ2, DTLZ3 AND DTLZ4 PROBLEMS. BEST PERFORMANCE IS SHOWN IN BOLD

Instance	NSGA/TD		NSGA-II
DTLZ1	Min	**3.17E-04**	9.91E-03
	Mean	**5.75E-04**	1.50E-02
	Worst	**1.10E-03**	3.20E-02
	Var.	**3.56E-08**	3.73E-05
DTLZ2	Min	**6.14E-04**	1.60E-02
	Mean	**1.88E-03**	1.93E-02
	Worst	**3.77E-03**	2.14E-02
	Var.	**5.07E-07**	1.35E-06
DTLZ3	Min	**3.61E-03**	2.57E-02
	Mean	**1.56E-02**	2.28E-01
	Worst	**3.31E-02**	1.07E+00
	Var.	**6.20E-05**	8.12E-02
DTLZ4	Min	**1.72E-03**	1.56E-02
	Mean	**2.28E-03**	1.83E-02
	Worst	**2.89E-03**	2.24E-02
	Var.	**7.99E-08**	1.22E-06

TABLE III
BEST, MEDIAN, AND WORST HV METRIC VALUES OBTAINED FOR NSGA/TD AND NSGA-II ON 3-OBJECTIVE DTLZ1, DTLZ2, DTLZ3 AND DTLZ4 PROBLEMS. BEST PERFORMANCE IS SHOWN IN BOLD

Instance	NSGA/TD		NSGA-II
DTLZ1	Best	**9.91E-02**	9.09E-02
	Mean	**9.90E-02**	8.78E-02
	Worst	**9.89E-02**	8.54E-02
	Var.	**1.29E-09**	3.31E-06
DTLZ2	Best	**4.14E-01**	3.49E-01
	Mean	**4.11E-01**	3.40E-01
	Worst	**4.09E-01**	3.33E-01
	Var.	**1.06E-06**	1.76E-05
DTLZ3	Best	**4.05E-01**	3.20E-01
	Mean	**3.76E-01**	1.65E-01
	Worst	**3.08E-01**	1.25E-01
	Var.	**3.25E-04**	9.04E-03
DTLZ4	Best	**4.13E-01**	3.57E-01
	Mean	**4.11E-01**	3.48E-01
	Worst	**4.09E-01**	3.35E-01
	Var.	**7.19E-07**	2.55E-05

TABLE IV
T-TEST VALUES OF IGD AMONG NSGA/TD AND NSGA-II

NSGA/TD vs. NSGA-II		
-	h-value	p-value
DTLZ1	**1.00E+00**	1.95E-18
DTLZ2	**1.00E+00**	2.70E-57
DTLZ3	**1.00E+00**	1.70E-04
DTLZ4	**1.00E+00**	1.01E-59

TABLE V
T-TEST VALUES OF HV AMONG NSGA/TD AND NSGA-II

NSGA/TD vs. NSGA-II		
-	h-value	p-value
DTLZ1	**1.00E+00**	2.80E-39
DTLZ2	**1.00E+00**	1.49E-63
DTLZ3	**1.00E+00**	6.00E-17
DTLZ4	**1.00E+00**	1.62E-56

TABLE VI
BEST, MEDIAN, AND WORST IGD VALUES OBTAINED FOR NSGA/TD, NSGA-III AND TWO VERSIONS OF MOEA/D ON 3-OBJECTIVE DTLZ1, DTLZ2, DTLZ3 AND DTLZ4 PROBLEM. BEST PERFORMANCE IS SHOWN IN BOLD

Problem	M	NSGA/TD		NSGA-III	MOEA/D-PBI	MOEA/D-TCH	MOEA/D-DE
DTLZ1	3	Min	4.78E-04	4.88E-04	**4.10E-04**	3.30E-02	5.47E-03
		Mean	**7.88E-04**	1.31E-03	1.50E-03	3.32E-02	1.78E-02
		Worst	**1.10E-03**	4.88E-03	4.47E-03	3.36E-02	3.39E-01
DTLZ2	3	Min	1.60E-03	1.26E-03	**5.43E-04**	7.50E-02	3.85E-02
		Mean	2.50E-03	1.36E-03	**6.41E-04**	7.57E-02	4.56E-02
		Worst	3.20E-03	2.11E-03	**8.01E-04**	7.66E-02	6.07E-02
DTLZ3	3	Min	1.70E-03	**9.75E-04**	9.77E-04	7.60E-02	5.61E-02
		Mean	3.50E-03	4.01E-03	**3.43E-03**	7.66E-02	1.44E-01
		Worst	1.28E-02	**6.67E-03**	9.11E-03	7.76E-02	8.89E-01
DTLZ4	3	Min	1.80E-03	**2.92E-04**	2.93E-01	2.17E-01	3.28E-02
		Mean	2.20E-03	**5.97E-04**	4.28E-01	3.72E-01	6.05E-02
		Worst	**2.90E-03**	4.29E-01	5.23E-01	4.42E-01	3.47E-01

in DTLZ1-DTLZ4. In DTLZ1-DTLZ4 test problems, the proposed algorithm has been significant better than NSGA-II and quite comparable with NSGA-III, MOEA/D-PBI, MOEA/D-TCH and MOEA/D-DE. In many real-world problem, the function of objectives are quite complex, maybe we can not obtain the decision variable that decided position of individual by solving eqution. In future work, it is a key point that generating individuals on faultage. Mechine Learning will provide some effective ways to solving the issue.

ACKNOWLEDGMENT

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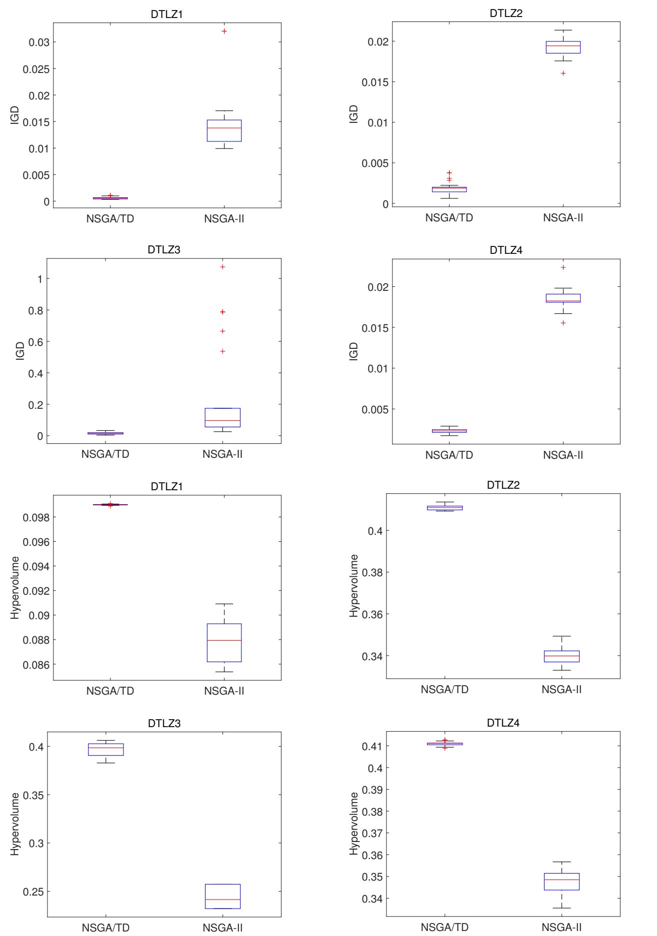

Fig. 3. IGD and HV metric value of boxing graph with NSGA-II and NSGA/TD for DTLZ1-DTLZ4

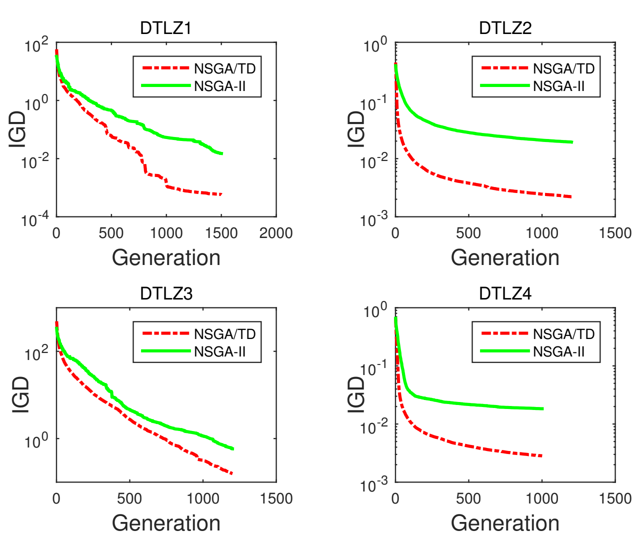

Fig. 4. variation of IGD metric value with NSGA-II and NSGA/TD for DTLZ1-DTLZ4

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